ON THE PERIODIC STRUCTURE OF A STATIONARY RAREFIED PLASMA

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In the investigation of boundary problems for a rarefied plasma the occurrence of stationary periodic solutions [1–3] has been noted on more than one occasion. Since the existence of such solutions leads to a finite change in certain plasma parameters for infinitesimal changes in other parameters, the region of periodic solutions is treated in a series of papers as an instability region [3, 4]. However, as far as the authors are aware, arbitrary assumptions have been made in existing papers regarding the distribution of charged particles. For example, in Bohm's article [3] a monovelocity model was proposed, and in the papers of Auer, Hurwitz, McIntyre and others, an arbitrary distribution of trapped particles was introduced.

Consequently, it is of interest to carry out a strict investigation of the question of whether spatial periodicity exists in a stationary rarefied plasma. The present paper finds criteria for the appearance of spatially periodic solutions for the self-consistent problem in the zero-th approximation in L/l (L is a characteristic dimension of the system, l is the mean free path of plasma particles).

For the sake of simplicity, in the calculations we shall confine ourselves to a one-dimensional boundary problem for a rarefied gas (composed of electrons, ions, and neutral particles) filling the space between two infinite parallel plates. We shall neglect the effect of magnetic fields. The potential distribution is described by Poisson's equation

$$\frac{d^2\varphi}{dx^2} = 4\pi e(\rho_e - \rho_i) \qquad \left(\rho_{e,i} = \int f_{e,i}\, d\mathbf{v}\right). \tag{1}$$

Here x is the coordinate reckoned from one of the electrons, $\rho_{e,i}$ is the electron or ion density, respectively, $f_{e,i}$ are the distribution functions of electrons and ions, respectively. The electron and ion distribution functions satisfy the kinetic equations

$$v_x \frac{\partial f_{e,i}}{\partial x} \pm \frac{e}{m_{e,i}} \frac{d\varphi}{dx} \frac{\partial f_{e,i}}{\partial v} = J_{e,i}. \tag{2}$$

Here $m_{e,i}$ are the electron and ion masses, respectively, $J_{e,i}$ are the collision integrals for electrons or ions. As boundary conditions we shall introduce given electrode potentials ($\varphi(0) = 0$, $\varphi(L) = \varphi_L$, L is the interelectrode distance) and equations describing the particle distributions on the electrodes.

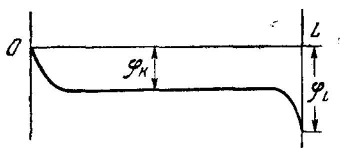

Fig. 1

It was shown in [5] that in the zero-th approximation in L/l a monotonic potential distribution in an interelectrode space filled by a rarefied gas of charged particles is possible for a given electrode potential difference only when the conditions

$$\rho \equiv \rho_e - \rho_i = 0, \quad d\varphi/dx = 0,$$
$$0 < d\rho/d\varphi < \infty \quad \text{for } \varphi = \varphi_k. \tag{3}$$

are fulfilled for a specific value of potential φ_k ($0 \leqslant \varphi_k < \varphi_L$).

Here the potential distribution has the appearance shown in Fig. 1: the potential is practically constant over the whole space and is equal to φ_k, with the exception of narrow layers in the neighborhood of the electrodes of width of the order of the Debye-Hückel radius λ_D, in which a sharp change of potential occurs from 0 to φ_k close to one electrode, and from φ_k to φ_L close to the other (here and in what follows, to be specific, we shall treat the case for $\varphi_L < 0$).

For gas parameters with $\varphi_k = \varphi_L$, the derivative $d\rho/d\varphi$ becomes infinite, and the last condition of inequality (3) is not fulfilled. The potential distribution in this limiting monotonic case is shown in Fig. 2: with the exception of a narrow boundary layer close to the coordinate origin the potential is strictly constant and equal to φ_k. Further change of the gas parameters leads to a nonmonotonic potential distribution. The conditions for it to be periodic are found below.

In order to do this we must examine Poisson's equation (1).

We shall assume that the plasma has a periodic structure (Fig. 3) in the neighborhood of the limiting monotonic configuration. In this case we must distinguish between particles trapped in potential wells and those which are not so trapped.

The integrals of motion for electrons ε_e and for ions ε_i satisfy the conditions

$$\varepsilon_e \geqslant -e\varphi_{\min}, \quad -e\varphi_{\min} \geqslant \varepsilon_e \geqslant -e\varphi_{\max},$$
$$(\varepsilon_e = {}^1/_2 m_e v_x^2 - e\varphi(x)),$$
$$\varepsilon_i \geqslant e\varphi_{\max}, \qquad e\varphi_{\min} \leqslant \varepsilon_i \leqslant e\varphi_{\max},$$
$$(\varepsilon_i = {}^1/_2 m_i v^2 + e\varphi(x)).$$

Here the first inequalities correspond to untrapped and the second to trapped particles, $\varphi_{\min}$ and $\varphi_{\max}$ correspond to the minimal and maximal values of potential in the interelectrode space. In the zero-th approximation in L/l under consideration the distribution functions depend only on the integrals of motion ε_e or ε_i.

Denoting the distribution function for untrapped particles with an index c, and for trapped particles with an index w, we may write down expressions for the electron and ion densities for $x \geq x_{\min}$ in the form

$$\rho_e(x) = \frac{1}{\sqrt{2m_e}}\Big[\int_{-e\varphi_{\min}}^{\infty} [f_{ec}^{+}(\varepsilon) + f_{ec}^{-}(\varepsilon)]\frac{d\varepsilon}{\sqrt{\varepsilon + e\varphi}} + \int_{-e\varphi}^{-e\varphi_{\min}} [f_{ew}^{+}(\varepsilon) + f_{ew}^{-}(\varepsilon)]\frac{d\varepsilon}{\sqrt{\varepsilon + e\varphi}}\Big],$$

$$\rho_i(x) = \frac{1}{\sqrt{2m_i}}\Big[\int_{e\varphi_{\max}}^{\infty} [f_{ic}^{+}(\varepsilon) + f_{ic}^{-}(\varepsilon)]\frac{d\varepsilon}{\sqrt{\varepsilon - e\varphi}} + \int_{e\varphi}^{e\varphi_{\max}} [f_{iw}^{+}(\varepsilon) + f_{iw}^{-}(\varepsilon)]\frac{d\varepsilon}{\sqrt{\varepsilon - e\varphi}}\Big]. \quad (4)$$

Here $f_{e,i}^{-}(\varepsilon)$ and $f_{e,i}^{+}(\varepsilon)$ are the distribution functions for particles with $v_x > 0$ and $v_x < 0$, respectively, in the zero-th approximation in L/l; they are assumed to be already integrated over v_y and v_z. Introducing the following symbols

$$f_{ec}^{+}(\varepsilon) + f_{ec}^{-}(\varepsilon) = f_{ec}(\varepsilon), \qquad f_{ic}^{+}(\varepsilon) + f_{ic}^{-}(\varepsilon) = f_{ic}(\varepsilon),$$
$$f_{ew}^{+}(\varepsilon) + f_{ew}^{-}(\varepsilon) = f_{ew}(\varepsilon), \qquad f_{iw}^{+}(\varepsilon) + f_{iw}^{-}(\varepsilon) = f_{iw}(\varepsilon), \quad (5)$$

and making an analytic continuation of f_{ec} and f_{ic} to the region of the integrals of motion ε_e, ε_i, corresponding to trapped particles, we write Eqs. (4) in the form

$$\rho_e = \frac{1}{\sqrt{2m_e}}\Big\{\int_{-e\varphi}^{\infty} f_{ec}(\varepsilon)\frac{d\varepsilon}{\sqrt{\varepsilon + e\varphi}} + \int_{-e\varphi}^{-e\varphi_{\min}} [f_{ew}(\varepsilon) - f_{ec}(\varepsilon)]\frac{d\varepsilon}{\sqrt{\varepsilon + e\varphi}}\Big\},$$

$$\rho_i = \frac{1}{\sqrt{2m_i}}\Big\{\int_{e\varphi}^{\infty} f_{ic}(\varepsilon)\frac{d\varepsilon}{\sqrt{\varepsilon - e\varphi}} + \int_{e\varphi}^{e\varphi_{\max}} [f_{iw}(\varepsilon) - f_{ic}(\varepsilon)]\frac{d\varepsilon}{\sqrt{\varepsilon - e\varphi}}\Big\}. \quad (6)$$

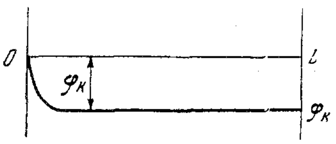

Fig. 2

In order to carry out further transformations, it is convenient to pass to the variables $t^2 = \varepsilon_{e,i} \pm e\varphi$, respectively, in Eqs. (6). Then

$$\rho_e = \frac{1}{\sqrt{m_e}}\Big\{\int_0^{\infty} f_{ec}(t^2 - e\varphi)\,dt + \int_0^{\tau_-} F_e(t^2 - e\varphi)\,dt\Big\}$$

$$(\tau_- = \sqrt{e(\varphi - \varphi_{\min})}), \quad (7)$$

$$\rho_i = \frac{1}{\sqrt{m_i}}\Big\{\int_0^{\infty} f_{ic}(t^2 + e\varphi)\,dt + \int_0^{\tau_+} F_i(t^2 + e\varphi)\,dt\Big\}$$

$$(\tau_+ = \sqrt{e(\varphi_{\max} - \varphi)}). \quad (7)\ \text{(cont'd)}$$

Here

$$F_{e,i} = f_{e,i,w} - f_{e,i,c}. \quad (8)$$

Assuming that the amplitude $\varphi_{\max} - \varphi_{\min}$ is small close to the limiting monotonic configuration, we may expand Eqs. (7) in a series in powers of $(\varphi_{\max} - \varphi_{\min})^{1/2}$. With an accuracy to half-integral powers of $\varphi_{\min} - \varphi$ we have

$$\rho_e = \frac{1}{\sqrt{m_e}} \times \Big\{\int_0^{\infty} f_{ec}(t^2 - e\varphi_{\min})\,dt + \sqrt{e(\varphi - \varphi_{\min})}\,F_e(-e\varphi_{\min})\Big\} \quad (9)$$

$$\rho_i = \frac{1}{\sqrt{m_i}}\Big\{\int_0^{\infty} f_{ic}(t^2 + e\varphi_{\min})\,dt + \sqrt{e(\varphi_{\max} - \varphi)}\,F_i(e\varphi_{\max})\Big\}.$$

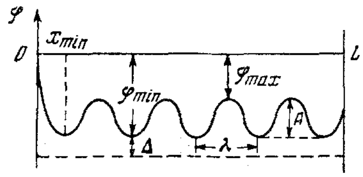

Fig. 3

Moreover, since the difference $\varphi - \varphi_k$ is small, where φ_k is the constant potential of the limiting monotonic configuration, we may expand ρ_e, ρ_i in powers of $\varphi_{\min} - \varphi_k$, assuming that the distribution functions have been analytically continued to the values $\varepsilon_e = -e\varphi_k$ and $\varepsilon_i = e\varphi_k$. With an accuracy to linear terms, we obtain

$$\rho_e = \frac{1}{\sqrt{m_e}}\Big\{\int_0^{\infty} f_{ec}(t^2 - e\varphi_k)\,dt + e(\varphi_k - \varphi_{\min})\int_0^{\infty}\Big[\frac{\partial f_{e,c}(\varepsilon)}{\partial\varepsilon}\Big]_{\varepsilon_1} dt + \sqrt{e(\varphi - \varphi_{\min})}\,F_e(-e\varphi_k)\Big\} \quad (\varepsilon_1 = t^2 - e\varphi_k),$$

$$\rho_i = \frac{1}{\sqrt{m_i}}\Big\{\int_0^{\infty} f_{ic}(t^2 + e\varphi_k)\,dt + e(\varphi_{\min} - \varphi_k)\int_0^{\infty}\Big[\frac{\partial f_{i,c}(\varepsilon)}{\partial\varepsilon}\Big]_{\varepsilon_2} dt + \sqrt{e(\varphi_{\max} - \varphi)}\,F_i(e\varphi_k)\Big\} \quad (\varepsilon_2 = t^2 + e\varphi_k). \quad (10)$$

We note that F_e and F_i represent the difference in values of the distribution functions for trapped and untrapped particles, respectively, at the boundary of the potential well. After setting ρ_e and ρ_i from Eqs. (10) in Poisson's equation, we find that it assumed the form

$$\frac{d^2\varphi}{dx^2} = a(\varphi_{\min} - \varphi_k) + b(\varphi - \varphi_{\min})^{1/2} + c(\varphi_{\max} - \varphi)^{1/2}. \quad (11)$$

Here

$$a = 4\pi e^2 \left\{ \frac{1}{\sqrt{m_e}} \int_0^\infty \left[\frac{df_{ec}}{d\varepsilon}\right]_{\varepsilon_1} dt - \frac{1}{\sqrt{m_i}} \int_0^\infty \left[\frac{df_{ic}}{d\varepsilon}\right]_{\varepsilon_2} dt \right\}, \quad (12)$$

$$b = \frac{4\pi e^{3/2}}{\sqrt{m_e}} F_e(-e\varphi_k), \qquad c = -\frac{4\pi e^{3/2}}{\sqrt{m_i}} F_i(e\varphi_k). \quad (13)$$

Since the first terms in Eqs. (10) are equal to $\rho_e(\varphi_k)$ and $\rho_i(\varphi_k)$, on being inserted in Poisson's equation they cancel out in accordance with the first condition of (3). Integrating Eq. (11) from $\varphi_{\min}$ to φ, we obtain

$$\tfrac{1}{2}\varphi'^2 = a\Delta(\varphi - \varphi_{\min}) + \tfrac{2}{3}[b(\varphi - \varphi_{\min})^{3/2} - c(\varphi_{\max} - \varphi)^{3/2} + c(\varphi_{\max} - \varphi_{\min})^{3/2}], \qquad (\Delta = \varphi_{\min} - \varphi_k) \quad (14)$$

Hence we find for the amplitude $A = \varphi_{\max} - \varphi_{\min}$ setting $\varphi = \varphi_{\max}$,

$$A^{1/2} = A_0^{1/2}\Delta,$$

$$A_0^{1/2} = -\frac{3}{2}\frac{a}{b+c} \quad (b + c \neq 0, \ \operatorname{sign} a = -\operatorname{sign}(b+c)). \quad (15)$$

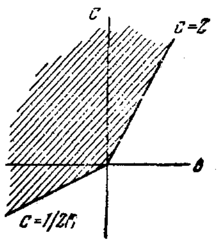

Fig. 4

The amplitude is proportional to the square of Δ, which confirms the validity of the assumption that the amplitude is small for small departures from the limiting monotonic configuration, and the validity of the expansion in small parameters made above. Integrating (14) from $\varphi_{\min}$ to $\varphi_{\max}$, we obtain the following expression for the wavelength λ:

$$\lambda = \lambda_0 \Delta^{1/2}, \quad \lambda_0 =$$

$$= \sqrt{2A_0} \int_0^1 \left\{ ay + \frac{2}{3} A_0^{1/2} [by^{3/2} + c[1 - (1-y)^{3/2}]] \right\}^{-1/2} dy. \quad (16)$$

Thus the wavelength is proportional to the root of Δ. Necessary conditions for the existence of periodic solutions are

$$\varphi''|_{\varphi=\varphi_{\min}} > 0, \quad \varphi''|_{\varphi=\varphi_{\max}} < 0.$$

Moreover, an odd number of points of inflection must lie in between a minimum and a maximum, i.e., the equation $p(\varphi) = 0$ must have an odd number of roots. The first two conditions lead to the inequalities

$$a\Delta + cA^{1/2} > 0,$$

$$a\Delta + bA^{1/2} < 0 \text{ or } \quad c > 2b, \ c > \tfrac{1}{2}b. \quad (17)$$

Here the value of the amplitude A from (15) has been employed. It is evident that the condition $c > b/2$ will be the stronger for $b < 0$, and condition* $c > 2b$ the stronger for $b > 0$.

In order to examine the roots of equation $p(\varphi) = 0$ it is convenient to write it in the form

$$1 - \frac{3(1-\gamma)u}{2} = \frac{3\gamma(1-u^2)^{1/2}}{2},$$

$$\left(u = \left(\frac{\varphi - \varphi_{\min}}{A}\right)^{1/2}, \quad \gamma = \frac{c}{b+c}\right) \quad (18)$$

with the help of (15).

The roots u_0 of this equation must satisfy the requirements

$$0 < u_0 < 1, \qquad \gamma\,[1 - \tfrac{3}{2}(1-\gamma)u_0] > 0.$$

The examination carried out showed that on fulfillment of the conditions $c > 2b$ and $c > b/2$ this equation has always only one root; thus the conditions mentioned for the zero-th approximation in L/l determine the boundary of the region for periodic solutions. This region is shaded in Fig. 4 ($b = 1$). We note that the position of the boundary is determined only by the relation between discontinuities in the electron and ion distribution functions at the boundary of the potential wells and is independent of Δ.

We shall ascertain the behavior of amplitude and wavelength close to these boundaries. As is clear from (15), as before, close to the boundary the amplitude A remains a small quantity of the order of $\Delta^{1/2}$. However, the wavelength begins to increase as we approach the boundary of this region. Rough estimates show that

$$\lambda_0 \sim \frac{1}{\sqrt{2/3 - \gamma}} \quad \text{as} \quad c \to \frac{b}{2}, \quad (\gamma \to 2/3),$$

$$\lambda_0 \sim \frac{1}{\sqrt{1/3 - \gamma}} \quad \text{as} \quad c \to 2b, \quad (\gamma \to 1/3).$$

The increase of wavelength may facilitate the experimental detection of periodicity.

The calculations which have been carried out were confined to terms of the order of half-integral powers of the amplitude. In the case where the coefficients b

*All the results obtained are immediately applicable to the case when the plasma potential is positive. To do this it suffices to take $\Delta = \varphi_k - \varphi_{\max}$ and reverse the sign of the coefficient a in (12). The criterion of periodicity expressed by $c > 2b$ and $c > b/2$.

and c are equal to zero (for example, if the distribution functions for untrapped particles pass continuously to the distribution functions for trapped particles), it becomes necessary to take into account linear terms, and Poisson's equation assumes the form

$$\varphi'' = a(\varphi - \varphi_k), \tag{19}$$

where a is given by Eq. (12). The solution of this equation has the form

$$\varphi - \varphi_m = \Delta(\operatorname{sh}\sqrt{ax} - 1) \tag{20}$$

for $a < 0$ the solution describes monochromatic periodicity with an amplitude $A \sim \Delta$ and a wavelength independent of Δ.

Thus, periodic solutions are allowed only for specific relations between the distribution functions in the zero-th approximation for trapped and untrapped particles. In order to verify whether these conditions are fulfilled in actual problems, the distribution functions for untrapped particles must be found by means of the boundary conditions and also the distribution of trapped particles with the help of the approximate kinetic equation [5]. The stability of these solutions and the influence of collisions in the stationary case still remain open questions.

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